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Passive Propagation of the Transmembrane Potential Into Homogeneous Cardiac Tissue at the Externally Applied Current

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Abstract—**Mathematical model of homogeneous cardiac tissue is used to show that externally applied current can create changes of transmembrane potential in long distances from stimulating electrodes.**

Keywords—**Defibrillation, bidomain model, transmembrane potential.**

I. Introduction

The principal difficulty of the modern defibrillation theory is that the changes in transmembrane potential (TMP) of homogeneous cardiac tissue induced by external electric field decay exponentially with smaller distance (space constant) from electrodes. This distance is about 1 mm for cardiac tissue, and determined by values of the mutual electrical capacities of the myocardium bioelectrolytes (membrane capacitance) and its electrical resistance. Two mechanisms for overcoming this problem are proposed [1,2].

In one spatial dimension, the mechanism of deep entry of TMP bases on the values of own electrical capacities of the external and internal cardiac domains [1]. TMP distribution consists of two parts. The first part decays exponentially with distance as usually. But second part decays exponentially much more poorly, like extracellular potential. It occurs because the space constant of the first part is determined by membrane capacitance, and space constant of the second part is determined by own capacities of the myocardium bioelectrolytes. And the value of own capacities is much less than value of membrane capacitance for cardiac tissue. The TMP amplitude essentially depends on value of the own electrical capacitance of intracellular domain.

In two spatial dimensions, the mechanism of deep entry of TMP bases on the intracellular and extracellular resistivity anisotropy ratios that must be unequal, as it is typical the case in cardiac tissue [2]. The model also predicts that this mechanism is the most effective for a given applied field strength when the electrode size and separation, or spatial features of the externally applied field are characterized by scalelengths that are approximately commensurate with two times heart wall thickness.

In this paper the proposed two mechanisms of deep entry of TMP into homogeneous cardiac tissue are integrated into one mathematical model. It is shown, that at high frequencies of stimulating current the first mechanism dominates, and at low frequencies - the second.

II. Model

A. Model equations

In two spatial dimension, passive TMP propagation is described by coupling equations, of the form:

$$\frac{\partial q_e}{\partial t} + \nabla \vec{I}_e = 0, \qquad \vec{I}_e = -G_e \nabla \varphi_e, \tag{1}$$

$$\frac{\partial q_i}{\partial t} + \nabla \vec{I}_i = 0, \qquad \vec{I}_i = -G_i \nabla \varphi_i, \tag{2}$$

$$q_e = (a_e + \beta c_m)\varphi_e - \beta c_m \varphi_i - \int_0^t I_{mi} dt, \tag{3}$$

$$q_i = -\beta c_m \varphi_e + (a_i + \beta c_m)\varphi_i + \int_0^t I_{mi} dt, \tag{4}$$

$$I_{mi} = \beta G_m (\varphi_i - \varphi_e), \tag{5}$$

$$G_e = \begin{bmatrix} g_{ex} & 0 \\ 0 & g_{ey} \end{bmatrix}, \quad G_i = \begin{bmatrix} g_{ix} & 0 \\ 0 & g_{iy} \end{bmatrix}, \tag{6}$$

$$I_{ex}(t, x = -L, y) = I_{ex}(t, x = L, y) = I_S(t, y), \tag{7}$$

$$I_{ey}(t, x = -L, y) = I_{ey}(t, x = L, y) = 0, \tag{8}$$

$$I_{ix}(t, x = -L, y) = I_{ix}(t, x = L, y) = 0, \tag{9}$$

$$I_{iy}(t, x = -L, y) = I_{iy}(t, x = L, y) = 0, \tag{10}$$

$$x \in [-L, L], y \in (-\infty, \infty) \tag{11}$$

where q_e, q_i are the density of electrical charges, $\vec{I}_e, \vec{I}_i$ are the flow density of electrical currents, G_e, G_i are the conductivity tensors, φ_e, φ_i are the electrical potentials, a_e, a_i are the own electrical capacities per unit volume of the myocardium bioelectrolytes for extracellular and intracellular domains, respectively. Besides c_m - mutual electrical capacitance of the myocardium bioelectrolytes (membrane capacitance per unit surface area of the cell membrane), β - surface-to-volume ratio for cell membrane, I_{mi} - density of ionic current of the membrane, G_m - fixed conductance per unit surface area of the cell membrane. $I_S(t, y)$ - flow density of external electrical current on the

myocardium boundaries, t - time, x, y - spatial coordinates, $2L$ - myocardium length. TMP is $U = \varphi_i - \varphi_e$.

Substituting (3)-(6) into (1)-(2), we obtain

$$\begin{aligned}&(1+\chi_e)\frac{\partial\varphi_e}{\partial t} - \frac{\partial\varphi_i}{\partial t} = \\ &= D_{ex}\frac{\partial^2\varphi_e}{\partial x^2} + D_{ey}\frac{\partial^2\varphi_e}{\partial y^2} + \frac{\varphi_i - \varphi_e}{\tau_m}\end{aligned}, \quad (12)$$

$$\begin{aligned}&-\frac{\partial\varphi_e}{\partial t} + (1+\chi_i)\frac{\partial\varphi_i}{\partial t} = \\ &= D_{ix}\frac{\partial^2\varphi_i}{\partial x^2} + D_{iy}\frac{\partial^2\varphi_i}{\partial y^2} - \frac{\varphi_i - \varphi_e}{\tau_m}\end{aligned}, \quad (13)$$

$$\chi_e = a_e/\beta c_m, \quad \chi_i = a_i/\beta c_m, \quad \tau_m = c_m/G_m, \quad (14)$$

$$D_{ex} = g_{ex}/\beta c_m, \quad D_{ey} = g_{ey}/\beta c_m, \quad (15)$$

$$D_{ix} = g_{ix}/\beta c_m, \quad D_{iy} = g_{iy}/\beta c_m. \quad (16)$$

B. Method

Let assume, that

$$I_S(t,y) = \exp(i\omega t)\widetilde{I}_S(y), \qquad \omega = 2\pi f, \quad (17)$$

where $\widetilde{I}_S(y)$ -spatial distribution of flow density of the external electrical current on the electrodes, f - current frequency. Besides,

$$\widetilde{I}_S(y) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\widetilde{I}_{SK}(k)\exp(iky)dk, \quad (18)$$

$$\varphi_e(t,x,y) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\widetilde{\varphi}_{ek}(x,k)\exp(i\omega t + iky)dk, \quad (19)$$

$$\varphi_i(t,x,y) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\widetilde{\varphi}_{ik}(x,k)\exp(i\omega t + iky)dk, \quad (20)$$

$$\widetilde{\varphi}_{ek}(x,k), \widetilde{\varphi}_{ik}(x,k) \sim \exp(-\lambda x). \quad (21)$$

In this case

$$\widetilde{\varphi}_{ek} = -\frac{\widetilde{I}_{SK}}{g_{ex}(\delta\lambda_1 + \lambda_3)}\left(\delta\frac{sh\lambda_1 x}{ch\lambda_1 L} + \frac{sh\lambda_3 x}{ch\lambda_3 L}\right), \quad (24)$$

$$\widetilde{\varphi}_{ik} = -\frac{\widetilde{I}_{SK}}{g_{ex}(\delta\lambda_1 + \lambda_3)}\left(p_1\delta\frac{sh\lambda_1 x}{ch\lambda_1 L} + p_3\frac{sh\lambda_3 x}{ch\lambda_3 L}\right), \quad (25)$$

$$p_1 = 1 + \chi_e\frac{i\omega\tau_m}{i\omega\tau_m + 1} - \frac{\tau_m}{i\omega\tau_m + 1}\left(\lambda_1^2 D_{ex} - k^2 D_{ey}\right), \quad (26)$$

$$p_3 = 1 + \chi_e\frac{i\omega\tau_m}{i\omega\tau_m + 1} - \frac{\tau_m}{i\omega\tau_m + 1}\left(\lambda_3^2 D_{ex} - k^2 D_{ey}\right), \quad (27)$$

$$\delta = -\lambda_3 p_3/\lambda_1 p_1. \quad (28)$$

And λ_1, λ_3 are roots of the characteristic equation:

$$\lambda^4 D_{ex}D_{ix} - \lambda^2 A + B = 0,$$

$$\begin{aligned}&A = D_{ex}\left(i\omega + \tau_m^{-1} + k^2 D_{iy}\right) + D_{ix}\left(i\omega + \tau_m^{-1} + k^2 D_{ey}\right) + \\ &+ i\omega\left(\chi_i D_{ex} + \chi_e D_{ix}\right)\end{aligned},$$

$$\begin{aligned}&B = k^4 D_{ey}D_{iy} + k^2\left(i\omega + \tau_m^{-1}\right)\left(D_{ey} + D_{iy}\right) + \\ &i\omega\left[\left(\chi_e + \chi_i\right)\left(i\omega + \tau_m^{-1}\right) + k^2\left(\chi_e D_{iy} + \chi_i D_{ey}\right)\right] + \chi_e\chi_i(i\omega)^2\end{aligned},$$

$$\lambda_1 = -\lambda_2, \qquad \lambda_3 = -\lambda_4$$

III. Results

Membrane thickness is a very small value, therefore $\chi_e, \chi_i << 1$. Let's assume additionally, that $k^2\tau_m D << 1$. In this case

$$\lambda_1^2 \approx \left(i\omega + \tau_m^{-1}\right)\left(D_{ex}^{-1} + D_{ix}^{-1}\right),$$

$$\lambda_3^2 \approx \left[i\omega\left(\chi_e + \chi_i\right) + k^2\left(D_{ey} + D_{iy}\right)\right]/\left(D_{ex} + D_{ey}\right).$$

Spatial distributions of electric potentials are characterized by two lengths $l_1 = 1/|\lambda_1|, l_3 = 1/[\lambda_3]$. For cardiac tissue l_1 is about 1 mm at the defibrillation conditions. Since $\chi_e, \chi_i << 1$, and $k^2\tau_m D << 1$, there is $l_3 >> l_1$, and $l_3 >> 1$ mm.

Let's find the approach to spatial distributions of electric potentials far from electrodes, i.e. under conditions $x \sim 0$, and $|\lambda_1 L| >> 1$, but $|\lambda_3 L| << 1$.

From (24)-(25), we obtain

$$\widetilde{U}_k = \widetilde{\varphi}_{ik} - \widetilde{\varphi}_{ek} \approx -(\Delta_1 + \Delta_2)\widetilde{\varphi}_{ek},$$

$$\Delta_1 = \frac{i\omega\tau_m}{i\omega\tau_m + 1}\left(\frac{\chi_i D_{ex} - \chi_e D_{ix}}{D_{ex} + D_{ix}}\right),$$

$$\Delta_2 = \frac{\tau_m k^2}{i\omega\tau_m + 1}\left(\frac{D_{ex}D_{iy} - D_{ey}D_{ix}}{D_{ex} + D_{ix}}\right).$$

For cardiac tissue $D_{ex} >> D_{ix}$. In this case

$$\Delta_1 \approx \chi_i \frac{i\omega\tau_m}{i\omega\tau_m + 1},$$

$$\Delta_2 \approx \frac{\tau_m k^2}{i\omega\tau_m + 1}\left(D_{iy} - D_{ix}\frac{D_{ey}}{D_{ex}}\right).$$

For low frequency, $\omega\tau_m << 1$:

$$\Delta_1 \approx 0,$$

$$\Delta_2 \approx \tau_m k^2\left(D_{iy} - D_{ix}\frac{D_{ey}}{D_{ex}}\right).$$

For higher frequency, $\omega\tau_m >> 1$:

$$\Delta_1 \approx \chi_i,$$

$$\Delta_2 \approx 0.$$

IV. Discussion

At the set distribution of density of external electric current to borders of a myocardium (1), expression (18)-(20),(24)-(28) defines time-spatial distributions of electric potentials in external and internal bioelectrolytes. In distances from stimulating electrodes the TMP is proportional to electrical potential of extracellular domain.

V. Conclusion

The TMP distribution consists of two parts. The first decays exponentially with distance as usual. But second part decays exponentially much more poorly, like extracellular potential. It occurs because the characteristic length of the first part is determined by membrane capacitance, and characteristic length of the second part is determined by own capacities of the myocardium bioelectrolytes and by wavenumber value in a cross-section direction to electrodes. And the value of own capacities is much less than value of membrane capacitance for cardiac tissue.

The TMP amplitude essentially depends on value of the own electrical capacitance of intracellular domain and intracellular and extracellular resistivity anisotropy ratios.

References

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